Using Algebra to Prove

Chapter 13

Lesson 13-1 If-Then Statements

BIG IDEA Statements that are of the form if . . . then are the basis of mathematical logic, so it is important to know how to determine if they are true or false.

If is one of the most important words in mathematics. (The words given, when, whenever, and suppose often have the same meaning.) The word if is often followed by the word then, which may or may not be written. The result is an if-then statement. Here are some examples.

1. If a bug is an insect, it has six legs.
2. Suppose a person likes outdoor football. Then the person will like arena football.
3. Every animal on land grows leaves.

In an if-then statement, the clause following if is called the antecedent. The clause following then is the consequent. Below we have underlined the antecedent once and the consequent twice.

If a bug is an insect, then it has six legs.

antecedent consequent

Some if-then statements are generalizations. A generalization is an if-then statement in which there is a variable in the antecedent and consequent. In Statements 1–3 above, the variable is not seen but each statement can be thought of as an if-then statement with a variable.

If \( B \) is an insect, (then) \( B \) has six legs.

If \( P \) likes outdoor football, then \( P \) will like arena football.

When Is an If-Then Statement True?

An if-then statement is true if its consequent is true for every value in the domain of the variables in its antecedent.

Mental Math

A 28-page newspaper consists of a News section, a Classifieds section, and a Sports section. The News section is twice as long as the Sports section, and the Classifieds section is half as long as the Sports section. How long is

a. the News section?
b. the Sports section?
c. the Classifieds section?
**Example 1**

**True or False**  If \( B \) is an insect, (then) \( B \) has six legs.

**Solution**  The statement is true because every insect has six legs. (Having six legs is one of the defining characteristics of insects.)

An if-then statement with a variable describes a pattern. As in any pattern, a value of the variable for which both the antecedent and consequent of an if-then statement is true is an instance of the statement. A beetle is an insect and a beetle has six legs. So a beetle is an instance of Statement 1.

Situations in which the antecedent is false do not affect whether an if-then statement is true. A dining room table might have six legs, but a table is not an insect. So a dining room table is not an instance of Statement 1.

A true if-then statement can be represented with a Venn diagram. The set of insects is placed inside the set of things with six legs.

**Example 2**

**True or False**  If \( x \) is a month of the year, then \( x \) has 31 days.

**Solution**  This statement is not true because there are some months that have fewer than 31 days.

If there is a value of the variable for which the antecedent of an if-then statement is true and the consequent is not true, then the if-then statement is false. This value is a counterexample to the statement. The month of November has only 30 days, so it is a counterexample to the if-then statement. One counterexample is enough to cause an if-then statement to be false.

Even though there are values of the variable for which the statement of Example 2 is true, because there is a counterexample, the generalization is false.
Example 3
Draw a Venn diagram for the statement: If \( G \) is a land animal, \( G \) grows leaves.

Solution
A tiger is a land animal and a tiger does not grow leaves, so the statement is false. In fact, no land animals grow leaves. So if \( G \) is a land animal, \( G \) never grows leaves. The Venn diagram has two circles that do not overlap.

If-Then Statements in Mathematics
If-then statements occur throughout mathematics because they clarify what is given information and what are conclusions.

GUIDED
Example 4
Fill in the blanks.

<table>
<thead>
<tr>
<th>If-Then Statement</th>
<th>True or False</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. If ( 3n + 5 = 65 ), then ( n = 20 ).</td>
<td>true</td>
<td>If ( 3n + 5 = 65 ), then ( 3n = ? ).</td>
</tr>
<tr>
<td></td>
<td></td>
<td>If ( 3n = ? ), then ( n = 20 ).</td>
</tr>
<tr>
<td>b. If a figure is a square, then it is a rectangle.</td>
<td>?</td>
<td>Definition of square: a square is ( ? ).</td>
</tr>
<tr>
<td>c. If ( x ) is a real number, then ( x^2 &gt; 0 ).</td>
<td>?</td>
<td>( 0 ) is a real number and ( 0^2 ) is not greater than ( 0 ).</td>
</tr>
<tr>
<td>d. If a quadrilateral has 3 right angles, then it is a square.</td>
<td>false</td>
<td>?</td>
</tr>
</tbody>
</table>

Counterexamples to the two false statements in Guided Example 4 can be pictured. For Statement c, graph \( y = x^2 \). Notice that the graph is not always above the \( x \)-axis, so it is not the case that \( x^2 > 0 \) for every value of \( x \). For Statement d, show a quadrilateral that has 3 right angles but is not a square.
Putting Statements into If-Then Form

Statements with the words “all,” “every,” and “no” can be rewritten in if-then form without changing their meaning.

**Example 5**

Rewrite each statement in if-then form.

<table>
<thead>
<tr>
<th>Statement</th>
<th>If-then Form with Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. Every whole number is a real number.</td>
<td>If ( x ) is a whole number, then ____? ____</td>
</tr>
<tr>
<td>b. All people born in the United States are U.S. citizens.</td>
<td>If ____? ____, then ( P ) is a U.S. citizen.</td>
</tr>
<tr>
<td>c. No power of a positive number is negative.</td>
<td>If ( p ) is a positive number and ( g ) is any real number, then ____? ____</td>
</tr>
</tbody>
</table>

**Questions**

**COVERING THE IDEAS**

In 1 and 2, identify the antecedent and the consequent of the if-then statement.

1. If the sun shines this afternoon, I will be happy.
2. The world would be a better place if people did not litter.

In 3 and 4, rewrite the statement as an if-then statement with a variable in it. Underline the antecedent once and the consequent twice.

3. Every integer greater than 1 is either a prime number or a product of prime numbers.
4. When a person drives over 35 miles per hour on that street, that person is speeding.

5. Explain why this if-then statement is true.
   If \( 3x + 16 > 10 \), then \( x > -2 \).

6. Explain why this if-then statement is false.
   If \( 3x + 16 > 10 \), then \( x > 0 \).

In 7–9, an antecedent is given. Complete the statement with two consequents that make it true.

7. If \( L \) and \( W \) are the length and width of a rectangle, then ____? ____.
8. If \( x^m \cdot x^n = x^{m+n} \), then ____? ____.
9. If \( n \) is divisible by 10, then ____? ____.
In 10–13, an if-then situation is given.
(a) Tell whether the statement is true or false.
(b) Draw a Venn diagram picturing the situation.
10. If a figure is a rectangle, then the figure is a square.
11. If you are in the United States and in Chattanooga, then you are in Tennessee.
12. If a number is an even number, then it is a prime number.
13. If \( x > 0 \), then \( -x > 0 \).

**APPLYING THE MATHEMATICS**

In 14–17, a false statement is given.
(a) Find a counterexample that shows it is false.
(b) Find the largest domain of the variable for which the statement is true.
14. If \( w \) is a real number, then \( w^3 \) is positive.
15. If \( t \) is a real number, then \( t^4 \) is positive.
16. If \( r \) is a real number, then \( 2r \geq r \).
17. If \( x \) is a real number and \( y = x^2 + x \), then \( y \geq 0 \).

In 18 and 19, draw a counterexample to the statement.
18. For all real numbers \( a \), the graph of \( y = ax^2 \) is a parabola.
19. A triangle cannot have two angles each with measure over 75°.

In 20, draw the following four true statements in one Venn diagram.
(a) Every rhombus is a parallelogram.
(b) Every rectangle is a parallelogram.
(c) If a figure is both a rhombus and a rectangle, then it is a square.
(d) Every square is both a rhombus and a rectangle.

In 21 and 22, rewrite the statement as an if-then statement and indicate why each statement is false.
21. All sentences have a subject, verb, and object.
22. Every president of the United States has served fewer than three terms.

**REVIEW**

23. **Skill Sequence** Factor each expression. (Lesson 12-4)
(a) \( a^2 - 36 \)
(b) \( n^2 - 5n - 36 \)
(c) \( x^2 - 5xy - 36y^2 \)
24. Fill in the Blank  Do this problem in your head. Because one thousand times one thousand equals one million, then 1,005 · 995 = ?. (Lesson 11-6)

25. Tickets to a hockey game cost $22 for adults and $16 for children. The total attendance at one game was 3,150 and the total revenue from ticket sales for the game was $66,258. How many of each kind of ticket were sold for the game?  (Lesson 10-5)

26. Write $2^{-3} + 4^{-3}$ as a simple fraction.  (Lesson 8-4)

27. Let $g(x) = \frac{3}{4}x - 7$.  (Lesson 7-6)
   a. Calculate $g(80)$.
   b. Calculate $g(-80)$.
   c. Describe the graph of $g$.

28. True or False  The slope of the line through $(x_1, y_1)$ and $(x_2, y_2)$ is the opposite of the slope of the line through $(x_2, y_2)$ and $(x_1, y_1)$.  (Lesson 6-6)

EXPLORATION

29. Consider this statement: No four points in a plane can all be the same distance from each other.
   a. Write the statement in if-then form.
   b. Is the statement true or false?
   c. If “in a plane” is deleted from the statement, show that the resulting statement is false.