BIG IDEA Some quadratic expressions arise from problems involving the areas of rectangles.

Recall that the degree of a monomial is the sum of the exponents of the variables in the monomial, and the degree of a polynomial is the highest degree among its monomial terms. For instance, the monomials $20x^5$ and $\frac{1}{3}ab^4$ have degree 5, while the polynomial $x^5 - x^6$ has degree 6. The word quadratic in today’s mathematics refers to expressions, equations, and functions that involve sums of constants and first and second powers of variables and no higher powers. That is, they are of degree 2. Specifically, if $a$, $b$, and $c$ are real numbers, $a \neq 0$, and $x$ is a variable,

$$ax^2 + bx + c$$

is the general quadratic expression in $x$,

$$ax^2 + bx + c = 0$$

is the general quadratic equation in $x$, and

$$f: x \rightarrow ax^2 + bx + c$$

is the general quadratic function in $x$.

We call $ax^2 + bx + c$ the standard form of a quadratic. In standard form, the powers of the variable are in decreasing order. Some quadratic expressions, equations, and functions are not in standard form, but they can be rewritten in standard form.

STOP QY1

There can also be quadratics in two or more variables. The general quadratic expression in two variables is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F.$$

Quadratic equations and quadratic functions in two variables are discussed in Chapter 12.

The simplest quadratic expression $x^2$ is the product of the simplest linear expressions $x$ and $x$. More generally, the product of any two linear expressions $ax + b$ and $cx + d$ is a quadratic expression for any real numbers $a$, $b$, $c$, and $d$, provided $a$ and $c$ are not zero. Because all area formulas involve the product of two lengths, they all involve quadratic expressions.

QY1 Write $3x - (5 - 2x^2)$ in standard form.
Lesson 6-1

Quadratic Expressions from Rectangles

**Example 1**

Hector and Francisca are remodeling their kitchen. They purchase a 6-foot by 2-foot pantry door, and are looking at different widths of molding to trim the door frame. If the trim is $w$ inches wide, write the total area of the door and trim in standard form.

**Solution**

Draw a picture. The door is surrounded by trim on 3 sides. The door with trim occupies a rectangle with length $72 + w$ inches and width $24 + 2w$ inches. The area of this rectangle is $(72 + w)(24 + 2w)$ square inches.

Use the Distributive Property to multiply $(72 + w)(24 + 2w)$. Think of $(72 + w)$ as a single number.

$$(72 + w)(24 + 2w) = (72 + w) \cdot 24 + (72 + w) \cdot 2w$$

Now apply the Distributive Property twice more.

$$= 72 \cdot 24 + w \cdot 24 + 72 \cdot 2w + w \cdot 2w$$

$$= 1728 + 24w + 144w + 2w^2$$

Arithmetic

$$= 1728 + 168w + 2w^2$$

Combine like terms.

In standard form, the total area of the door and the trim is $2w^2 + 168w + 1728$ square inches.

**Check**

Use a CAS to expand the expression.

**QY2**

**Quadratic Expressions from Squares**

The expression $x + y$ is an example of a binomial. In general, a binomial is an expression with two terms. The square of a binomial can be thought of as the area of a square whose side length is the binomial.

**Example 2**

Write the area of the square with sides of length $x + y$ in standard form.

**Solution 1**

Draw a picture of the square. Notice that its area is the sum of four smaller areas: a square of area $x^2$, two rectangles, each with area $xy$, and a square with area $y^2$. So, the area of the original square is $x^2 + 2xy + y^2$.

*(continued on next page)*
Solution 2  The area of a square with side \( x + y \) is \((x + y)^2\).

Rewrite \((x + y)^2\).

\[
(x + y)^2 = (x + y)(x + y) \quad \text{Definition of second power}
\]
\[
= (x + y)x + (x + y)y \quad \text{Distributive Property}
\]
\[
= x^2 + xy + xy + y^2 \quad \text{Distributive Property}
\]
\[
= x^2 + 2xy + y^2 \quad \text{Commutative Property of Multiplication and Distributive Property}
\]

The area of the square is \(x^2 + 2xy + y^2\).

When a linear expression is multiplied by itself, or squared, the result is a quadratic expression. In Example 2, the linear expression \( x + y \) is squared. You can also say it is “taken to the 2nd power.” Writing this power as a quadratic expression is called expanding the power. Squares of binomials occur so often that their expansions are identified as a theorem.

**Binomial Square Theorem**

For all real numbers \( x \) and \( y \),

\[
(x + y)^2 = x^2 + 2xy + y^2 \quad \text{and} \quad (x - y)^2 = x^2 - 2xy + y^2.
\]

Solution 2 of Example 2 provides the proof of the first part of this theorem. You are asked to complete the second part of this proof in Question 13. The Binomial Square Theorem is so useful that you will want to be able to apply it automatically.

**Example 3**

A city wants to cover the seating area around a circular fountain with mosaic tiles. The radius of the fountain, including \( s \) feet for seating, is 15 feet.

a. Write a quadratic expression in standard form for the area of the fountain, not including the seating area.

b. How many square feet are in the seating area in terms of \( s \)?

**Solution**

a. Draw a picture. The radius of the fountain without seating is \( _? \) ft. So, the fountain area without seating is \( \pi ( _? )^2 \) ft \(^2\).

To expand \(( _? - _? )^2\), use the Binomial Square Theorem with \( x = _? \) and \( y = s \).
\[(\ ? - s)^2 = \ ?^2 - 2 \cdot \ ? \cdot s + s^2\]
\[= s^2 - \ ? \cdot s + \ ?\]

So the area of the fountain not including seating is \(\pi(\ ?^2)\) or \(\pi \ ? - 30\pi s + 225\pi\) ft².

b. The seating area is the total area minus the fountain area without seating.
\[15^2\pi - (\ ?)^2\pi = 225\pi - (\ ? - \ ? + \ ?)\pi\]
\[= \ ? - \ ?\]

So the area of the seating area is \(\ ?\) ft².

Questions

**COVERING THE IDEAS**

1. **Multiple Choice** Which is not a quadratic equation? Explain your answer.
   - A \( y = \frac{x^2}{6}\)
   - B \( y = 2(x - 4)\)
   - C \( \frac{x^2}{4} - \frac{y^2}{9} = 25\)
   - D \( y = (x + 3)(2x - 5)\)

2. Is \(x^2 + \sqrt{7}\) a quadratic expression? Explain your answer.

3. Name two geometric figures for which quadratic expressions describe their area.

4. A door is 7 feet high and 30 inches wide, with trim of \(w\) inches wide on three sides of the door frame. Write the total area (in square inches) of the door and trim in standard form.

In 5–8, the product of two linear expressions is given.
   a. Rewrite the product as a single polynomial.
   b. Check your results using the `expand` command on a CAS.
5. \((3x + 5y)(2x + 7y)\)
6. \((x - 3)(x + 2)\)
7. \((1 - 2y)(2 - 3y)\)
8. \((6 + b)(2 - b)\)

In 9–11, expand the square of the binomial.
9. \((10 + 3)^2\)
10. \((d - 6)^2\)
11. \((p + w)^2\)

12. Draw a geometric diagram of the expansion of \((x + 5)^2\).
13. Prove the second part of the Binomial Square Theorem.

In 14 and 15, rewrite the expression in the form \(ax^2 + bx + c\).
14. \(\left(3t - \frac{1}{3}\right)^2\)
15. \((1 - p)^2\)
16. Refer to the quadratic expression in Example 1. Graph
   \[ y = (72 + x)(24 + 2x) \text{ and } y = 2x^2 + 168x + 1728 \]
   in the window \( \{x \mid -160 \leq x \leq 160\}, \{y \mid -2250 \leq y \leq 5750\} \).
   a. Trace the graph and toggle between the two graphs for at least three different values of \( x \). What do you notice about the ordered pairs when you switch between the graphs?
   b. Look at a table of values for the two functions. Does the table support your observation from Part a?
   c. Based on your results in Parts a and b, what do you conclude about the two equations you graphed?

17. Suppose a rectangular swimming pool with dimensions of 100 feet by 12 feet is surrounded by a walkway of width \( w \).
   a. Write a quadratic expression in standard form that gives the area of the pool and walkway together.
   b. Write an expression that gives the area of the walkway only.

**APPLYING THE MATHEMATICS**

In 18 and 19, rewrite the expression in the form \( ax^2 + bxy + cy^2 \).

18. \((2a + 3b)^2\)

19. \((2t - \frac{k}{3})^2\)

20. Refer to Example 3. If \( s = 3 \) feet and 12 mosaic tiles cover one square foot, how many tiles would be needed to cover the seating area around the fountain?

21. Certain ceramic tiles are 4 inches by 8 inches and are separated by grout seams that are \( x \) inches wide.
   a. Write a quadratic expression in standard form for the area covered by each tile and its share of the grout. (The grout in each seam is shared by two tiles, so each tile’s share is only half the grout in each seam.)
   b. If the grout seams are \( \frac{1}{2} \) inch wide, approximately how many tiles will it take to cover a 5-foot by 10-foot wall?
   c. What percent of the wall in Part b is grout?
   d. If the grout seams are \( \frac{3}{4} \)-inch wide, what percent of the wall will be grout?

22. a. Expand \((x - y)^2 - (x + y)^2\) on a CAS.
   b. Verify the CAS solution by using the Binomial Square Theorem.

In 23 and 24, find \( h \) so that the given equation is true.

23. \( x^2 + 20x + 100 = (x + h)^2 \)

24. \( x^2 - hx + h = (x - \sqrt{h})^2 \)
REVIEW

In 25 and 26, use the following data about the United States National Parks. (Lessons 3–5, 2–4)

<table>
<thead>
<tr>
<th>Year</th>
<th>Number of Recreation Visits (millions)</th>
<th>Federal Appropriations (billions of $)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1999</td>
<td>287.1</td>
<td>2.030</td>
</tr>
<tr>
<td>2000</td>
<td>285.9</td>
<td>2.112</td>
</tr>
<tr>
<td>2001</td>
<td>279.9</td>
<td>2.568</td>
</tr>
<tr>
<td>2002</td>
<td>277.3</td>
<td>2.654</td>
</tr>
<tr>
<td>2003</td>
<td>266.1</td>
<td>2.546</td>
</tr>
</tbody>
</table>

Source: National Park Service

25. Find the rate of change from 1999 to 2003 for each of the following:
   a. number of recreation visits
   b. federal appropriations

26. a. Find an equation for the line of best fit describing the number of recreation visits as a function of the year using only the years 1999, 2001, and 2003. Let \( x \) be the number of years after 1999.
   b. How well does the line of best fit predict the value for 2002?

27. a. Draw \( \triangle ABC \) with vertices \( A = (0, 0), B = (1, 1), \) and \( C = (2, 4) \).
   b. Draw \( \triangle A'B'C' \), its image under the transformation \( (x, y) \rightarrow (x - 5, y + 2) \).
   c. Describe the effect of this transformation on \( \triangle ABC \). (Lesson 4-10)

28. If \( m \) varies inversely as \( t^2 \), and \( m = 14 \) when \( t = 2.5 \), find the value of \( m \) when \( t = 7 \). (Lesson 2–2)

EXPLORATION

29. Doorway trim comes in various widths.
   a. Find out the prices of at least three different widths of doorway trim.
   b. Suppose you want to frame a 7-foot by 3-foot door. Find the area of the door moldings for each of the three different sizes you have found.
   c. How much will it cost to frame the door with each size of molding?
   d. Does the cost in Part b vary directly as the area of the molding? If so, how?

QY ANSWERS

1. \( 2x^2 + 3x - 5 \)
2. \( a = 2, b = 168, c = 1728, \) and \( x = w \)
3. \( (3x)^2 - 2(3x)y + y^2 = 9x^2 - 6xy + y^2 \)